Math 259A Lecture 9 Notes

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1 Kaplansky's Theorem and Polar Decomposition

1.1 Kaplansky's theorem, general case

Let's finish the proof of Kaplansky' theorem.

Theorem 1.1 (Kaplansky, late 50s)**.** *Let $M \subseteq \mathcal{B}(H)$ be a von Neumann algebra, and let $M_0 \subseteq M$ be a SO-dense *-algebra. Then $(\overline{M_0}^{\text{so}})_1 = (M)_1$. Moreso, $(\overline{M_{0,h}}^{\text{so}})_1 = (M_h)_1$ and $(\overline{M_{0,+}}^{\text{so}})_1 = (M_+)_1$.*

In other words, if $x \in M$, there exist $x_i \in M_0$ such that $\|x_i\| \leq \|x\|$ and $x_i \xrightarrow{\text{so}} x$. We have shown this in the case where $x = x^* \in (M_1)$. Let's extend it to the non-self-adjoint case.

Proof. If $x \in (M_+)_1$, then there exist $y_i \in (M_{0,h})_1$ such that $y_i \xrightarrow{\text{so}} \sqrt{x}$. But then $y_i^2 \xrightarrow{\text{so}} (\sqrt{x})^2 = x$; this is because

$$(y_i^2 - y^2)\xi = y_i(y_i - y)\xi + (y_i - y)(y\xi).$$

and $y_i \xrightarrow{\text{so}} y$.

To deal with general $x \in (M)_1$, consider the *-algebra of matrices $M_2(M) \subseteq M_2(\mathcal{B}(H)) = \mathcal{B}(H \oplus H)$. This algebra of matrices is SO-closed, so it is a von Neumann algebra. Moreover, $M_2(M_0)$ is SO dense in $M_2(M)$. By the first part, the operator

$$Y = \begin{bmatrix} 0 & x \\ x^* & 0 \end{bmatrix} \in (M_2(M))_1.$$

The norm of Y is 1 because $Y^* = Y$, and Y^*Y is diagonal with norm $\|x\|^2$. So there exist $Y_i \in (M_2(M_0)_h)_1$ such that $Y_i \xrightarrow{\text{so}} Y$. Since $\|Y_i\| \leq 1$, $\|[Y_i]_{1,2}\| \leq 1$. So we get $[Y_i]_{1,2} \xrightarrow{\text{so}} [Y_i]_{1,2} = x$. $\square$

Corollary 1.1. *Let $M \subseteq \mathcal{B}(H)$ be a *-algebra with unit. The following are equivalent:*

1. *M is a von Neumann algebra (i.e. is WO-closed)*

2. M is ultraweak-closed.

3. $(M)_1$ is ultraweak compact.

Proof. (1) $\implies$ (3): If M is a von Neumann algebra, then $M = (M_*)^*$, so 3 follows by the Banach-Alaoglu theorem.

(3) $\implies$ (1): This follows from Kaplansky's theorem. □

1.2 Polar decomposition

Definition 1.1. If $x \in \mathcal{B}(H)$, the **left support** $\ell(x)$ is the orthogonal projection onto $[xH]$, and the **right support** $r(x)$ is the orthogonal projection onto $(\ker x)^\perp = \overline{\operatorname{im} x^*}$.

Proposition 1.1. *The left and right support satisfy the following:*

1. $\ell(x)$ is the smallest projection $e \in B(H)$ such that $ex = x$.

2. $r(x)$ is the smallest projection $f \in B(H)$ such that $xf = x$.

So if $x = x^*$, then $\ell(x) = r(x)$.

Definition 1.2. If $x \in \mathcal{B}(H)$ is self adjoint, then $s(x) = \ell(x) = r(x)$ is called the **support** of x.

Recall that a **partial isometry** v is an element such that v^*v and vv^* are projections.[1]

Proposition 1.2. *If $v \in \mathcal{B}(H)$ is a partial isometry, then $\ell(v) = vv^*$ and $r(v) = v = v^*v$.*

Theorem 1.2 (Polar decomposition)**.** *Let $x \in B(H)$. There exist a unique $a \in B(H)_+$ and partial isometry $v \in B(H)$ such that $x = va$ and $v^*v = s(a)$.*

Remark 1.1. This is analogous to the fact that if $\alpha \in \mathbb{C}$, we can express $\alpha = \frac{\alpha}{|\alpha|}|\alpha|$.

Proof. Observe that if $x = va$, then $x^*x = av^*va = a^2$. So $a = \sqrt{x^*x}$.

How should we define v? If $\xi \in r(x)(H)$, then $x\xi = va\xi$ with $\|x\xi\|^2 = \langle x^*x\xi, \xi\rangle = \left\langle a^2\xi, \xi\right\rangle = \|a\xi\|^2$. So we define $v(a\xi) := x\xi$ for $a\xi \in s(a)H$ and $v(\eta) := 0$ if $\eta \perp s(a)(H)$. So v is a partial isometry on H.

For uniqueness, we saw that we must have $a = \sqrt{x^*x}$. If, in addition, $v^*v = s(x)$, then $va\xi = x\xi$. So this choice is forced upon us. □

[1] If one of these is a projection, so is the other.